Analytic torsion and dynamical zeta functions

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Dynamical zeta functions of **Ruelle** and **Selberg** are

- functions of a complex variable s
- tools to count the periodic orbits of a dynamical system, geometrically:
- attached to the **geodesic flow** on the unit sphere bundle $S(X)$ over a hyperbolic manifold X.

The dynamical zeta functions are represented by **Euler-type products**.

- Analogy to the **Riemann zeta function**

$$\left.\begin{aligned}\zeta(s) = \prod_{p=\text{prime}} (1-p^{-s})^{-1} \\ \mathrm{Re}(s) > 1\end{aligned}\right\} \leftrightarrow \left.\begin{aligned}R(s) = \prod_{\gamma=\text{prime}} (1-e^{-sl(\gamma)})^{-1} \\ \mathrm{Re}(s) > 1\end{aligned}\right\}$$

•Why are they interesting?

•Meromorphic continuation gives relations to spectral invariants such as

- eta invariant of Dirac-type operators
- analytic torsion

•Algebraic and geometric setting

- $G = \mathrm{SO}^0(d,1)$, $K = \mathrm{SO}(d)$, $d = 2n+1, n \in \mathbb{N}_>$
- $\widetilde{X} := G/K \cong \mathbb{H}^d$ using the Killing form
- Γ discrete, cocompact, torsion-free subgroup of G
- $X = \Gamma \backslash \widetilde{X}$ is a d-dimensional locally symmetric compact hyperbolic manifold

•Fix notation

- $\mathfrak{g}$ =Lie algebra of G
- $\mathfrak{k}$ =Lie algebra of K
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, the Cartan decomposition of $\mathfrak{g}$
- $\mathfrak{a}$ = a maximal abelian subalgebra of $\mathfrak{p}$
- A subgroup of G with Lie algebra $\mathfrak{a}$
- $M := \mathrm{Centr}_K(A)$

•The dynamical zeta functions are associated with the **geodesic flow** on $S(X) \cong \Gamma\backslash G/M$.
•Γ cocompact --→ every $\gamma \in \Gamma$, with $\gamma \neq e$ is hyperbolic.
Then:

Lemma (Wallach '76 ([Wal76]))

Let $\gamma \in \Gamma$, with $\gamma \neq e$. There exist a $g \in G$, a $m_\gamma \in M$, and an $a_\gamma \in A$, such that $g^{-1}\gamma g = m_\gamma a_\gamma$. The element m_γ is determined up to conjugacy in M, and the element a_γ depends only on γ.

Representation theory is involved $\dashrightarrow$ modern consideration of the dynamical zeta functions

Let $\chi\colon \Gamma \to \mathrm{GL}(V_\chi)$ be a finite dimensional representation of Γ and $\sigma \in \widehat{M}$.

Definition (Twisted Selberg zeta function)

$$Z(s;\sigma,\chi) := \prod_{\substack{[\gamma]\neq e \\ [\gamma]\ \text{prime}}} \prod_{k=0}^{\infty} \det\Big(\mathrm{Id} - (\chi(\gamma)\otimes\sigma(m_\gamma)\otimes S^k(\mathrm{Ad}(m_\gamma a_\gamma)|_{\overline{\mathfrak{n}}}))e^{-(s+|\rho|)|l(\gamma)}\Big),$$

for $\mathrm{Re}(s) > r$, r positive constant.

Definition (twisted Ruelle zeta function)

$$R(s;\sigma,\chi) := \prod_{\substack{[\gamma]\neq e \\ [\gamma]\ \text{prime}}} \det\left(\mathrm{Id} - (\chi(\gamma)\otimes\sigma(m_\gamma))e^{-sl(\gamma)}\right)^{(-1)^{d-1}},$$

for $\mathrm{Re}(s) > c$, c positive constant.

- The products run over the prime conjugacy classes $[\gamma]$ of Γ, which corresponds to the prime closed geodesics on X of length $l(\gamma)$.

Related work

1. Fried ([Fri95]) proved that the Ruelle zeta function associated with the geodesic flow on the unit sphere bundle of a closed manifold with C^{ω}-Riemannian metric of negative curvature admits a meromorphic continuation to $\mathbb{C}$.
2. Bunke and Olbrich ([BO95]) proved the meromorphic continuation of the twisted dynamical zeta functions for unitary representations of Γ for all locally symmetric spaces of real rank one ⇢ functional equations for them.
3. Wotzke ([Wot08]) proved the meromorphic continuation of the Ruelle zeta function for a compact odd dimensional hyperbolic manifold ⇢ analytic at $s = 0$ and related to the Ray-Singer analytic torsion.

Arbitrary representations of Γ on a f.d. vector space

$$\chi : \Gamma \to \mathrm{GL}(V_\chi)$$

- **Problem:** there is no Hermitian metric h^χ, which is compatible with the flat connection on the flat vector bundle $\dashrightarrow$ we deal with non self-adjoint Laplace and Dirac operators.
- **Solution:** Twisted Bochner-Laplace operator $\Delta^{\sharp}_{\tau,\chi}$ acting on smooth sections of twisted vector bundles $E_\tau \otimes E_\chi$.
- Locally: for $U \subset X$ open

$$\Delta^{\sharp}_{\tau,\chi}|_U = (\Delta_\tau \otimes \mathrm{Id}_{V_\chi})|_U$$

Spectral properties of the twisted Bochner-Laplace operator

- $\Delta^{\sharp}_{\tau,\chi}$ has the same principal symbol as $\Delta_{\tau} \otimes \mathrm{Id}_{V_{\chi}}$ ⇢ it is an elliptic operator ⇢ nice spectral properties ⇢ its spectrum is discrete and contained in a translate of a positive cone in $\mathbb{C}$.
- Consider the corresponding heat semi-group $e^{-t\Delta^{\sharp}_{\tau,\chi}}$. It is an integral operator with smooth kernel.
- Consider the trace of the operator $e^{-t\Delta^{\sharp}_{\tau,\chi}}$ and derive a trace formula.

Results

- **Main tool: Trace formula for the operator $e^{-tA^{\sharp}_{\chi}(\sigma)}$**

Theorem

For every $\sigma \in \widehat{M}$ we have

$$\mathrm{Tr}(e^{-tA^{\sharp}_{\chi}(\sigma)}) = \dim(V_{\chi})\,\mathrm{Vol}(X)\int_{\mathbb{R}} e^{-t\lambda^2} P_{\sigma}(i\lambda)d\lambda + \sum_{[\gamma]\neq e} \frac{l(\gamma)}{n_{\Gamma}(\gamma)} L_{sym}(\gamma;\sigma)\frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}},$$

where

$$L_{sym}(\gamma;\sigma) = \frac{\mathrm{tr}(\sigma(m_{\gamma})\otimes\chi(\gamma))e^{-|\rho|l(\gamma)}}{\det(\mathrm{Id}-\mathrm{Ad}(m_{\gamma}a_{\gamma})_{\overline{\mathfrak{n}}})}.$$

Results

- **Meromorphic continuation for arbitrary representations of Γ**

Theorem

The Selberg zeta function $Z(s;\sigma,\chi)$ admits a meromorphic continuation to the whole complex plane $\mathbb{C}$. The set of the singularities equals $\{s_k^{\pm}=\pm i\sqrt{t_k} : t_k \in \mathrm{spec}(A^{\sharp}_{\chi}(\sigma)), k\in\mathbb{N}\}$. The orders of the singularities are equal to $m(t_k)$, where $m(t_k)\in\mathbb{N}$ denotes the algebraic multiplicity of the eigenvalue t_k. For $t_0=0$, the order of the singularity s_0 is equal to $2m(0)$.

Theorem

For every $\sigma\in\widehat{M}$, the Ruelle zeta function $R(s;\sigma,\chi)$ admits a meromorphic continuation to the whole complex plane $\mathbb{C}$.

Results

- **Functional equations**

Theorem

The Selberg zeta function $Z(s;\sigma,\chi)$ satisfies the functional equation

$$\frac{Z(s;\sigma,\chi)}{Z(-s;\sigma,\chi)} = \exp\Big(-4\pi \dim(V_\chi)\,\mathrm{Vol}(X)\int_0^s P_\sigma(r)dr\Big),$$

where P_σ denotes the Plancherel polynomial associated with $\sigma \in \widehat{M}$.

Results

- **Functional equations**

Theorem

The super Ruelle zeta function associated with a non-Weyl invariant representation $\sigma \in \widehat{M}$ satisfies the functional equation

$$R^s(s;\sigma,\chi)R^s(-s;\sigma,\chi) = e^{2i\pi\eta(D^{\sharp}_{p,\chi}(\sigma))},$$

where $\eta(D^{\sharp}_{p,\chi}(\sigma))$ denotes the eta invariant of the twisted Dirac operator $D^{\sharp}_{p,\chi}(\sigma)$.

Results

- **Determinant formula**

Proposition

The Ruelle zeta function has the representation

$$R(s;\sigma,\chi)=\prod_{p=0}^{d-1}\det(A^{\sharp}_{\chi}(\sigma_p\otimes\sigma)+(s+\rho-\lambda)^2)^{(-1)^p}$$

$$\exp\Big(-2\pi(d+1)\dim(V_\chi)\dim(V_\sigma)\operatorname{Vol}(X)s\Big),$$

where σ_p denotes the p-th exterior power of the standard representation of M and $\det(A^{\sharp}_{\chi}(\sigma_p\otimes\sigma)+(s+\rho-\lambda)^2)$ *the regularized determinant of the operator* $A^{\sharp}_{\chi}(\sigma_p\otimes\sigma)+(s+\rho-\lambda)^2$.

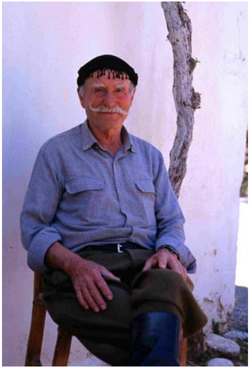
- **Discussion**
  The determinant formula motivates us to consider the Ruelle zeta function at zero as a candidate for the analytic torsion associated with the representation χ of Γ. However, in our setting we will have to introduce the refined analytic torsion. The expression of the refined analytic torsion involves the exponential of the eta invariant, as in the functional equations for the Ruelle zeta function. On the other hand, we consider an acyclic representation χ of Γ, but we can not apply the Hodge theory and hence the regularity of the Ruelle zeta function at zero can not be implied.

Future projects

- Prove that the Ruelle zeta function is regular at 0 and equals the refined analytic torsion as it is introduced by Braverman and Kappeler ([BK08]). This is rather a complex refinement of the Ray-Singer analytic torsion. Key-idea: Consider representation of Γ, which is non-unitary, but very special one: If $\tau : G \to \mathrm{GL}(V)$ is a finite dimensional complex representation of G, let $\tau|_\Gamma$ be the restriction to Γ.
- Even-dimensional case: The problem is the presence of the discrete series representation of G. The Plancherel measure is more complicated.
- For a compact hyperbolic surface, this is a joint work with J. Bajpai (MPI, Bonn).

Future projects

- Consider all locally symmetric spaces of real rank 1:
  - the complex hyperbolic d-dimensional case. Lie group $\mathrm{SU}(d,1)$;
  - the quaternionic hyperbolic d-dimensional space. Lie group $\mathrm{Sp}(d,1)$;
  - the Cayley upper half plane. Lie group F_4^{-20}.



Thank you for your attention!

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